Long range correlations in the non-equilibrium quantum relaxation of a spin chain

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We consider the non-stationary quantum relaxation of the Ising spin chain in a transverse field of strength h. Starting from a homogeneously magnetized initial state the system approaches a stationary state by a process possessing quasi long range correlations in time and space, independent of the value of h. In particular the system exhibits aging (or lack of time translational invariance on intermediate time scales) although no indications of coarsening are present.

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Non-equilibrium dynamical properties of quantum systems have been of interest recently, experimentally and theoretically. Measurements on magnetic relaxation at low-temperatures show deviations from the classical exponential decay [1], which was explained by the effect of quantum tunneling. On the theoretical side, among others, integrable [2] and non-integrable models [3] were studied in the presence of energy or magnetic currents, as well as the phenomena of quantum aging in systems with long-range [4] and short-range interactions [5].

Here we pose a different question: Consider a quantum mechanical interacting many body system described by a Hamilton operator $\hat{H}$ without any coupling to an external bath, which means that the system is closed. Suppose the system is prepared in a specific state $|\psi_0\rangle$ at time $t=0$, which is *not* an eigenstate of the Hamiltonian $\hat{H}$. Then we are interested in the natural quantum dynamical evolution of this state which is described by the Schrödinger equation and is formally given by

$$|\psi(t)\rangle = \exp\left(-\frac{i}{\hbar}\hat{H}t\right)|\psi_0\rangle \; . \qquad (1)$$

Obviously the energy $E=\langle\psi_0|\hat{H}|\psi_0\rangle$ is conserved. In particular we want to study the time evolution of the expectation value $A(t)$ of an observable $\hat{A}$ or the two-time correlation function $C_{AB}(t_1,t_2)$ of two observables $\hat{A}$ and $\hat{B}$, defined by

$$\begin{aligned} A(t) &= \langle\psi_0|\hat{A}_H(t)|\psi_0\rangle \\ C_{AB}(t_1,t_2) &= \langle\psi_0|\{\hat{A}_H(t_1)\hat{B}_H(t_2)\}_S|\psi_0\rangle \; , \end{aligned} \qquad (2)$$

where $\hat{A}_H(t)=\exp(+i\hat{H}t)\hat{A}\exp(-i\hat{H}t)$ is the operator $\hat{A}$ in the Heisenberg picture (with $\hbar$ set to unity) and $\{\hat{A}\hat{B}\}_S=1/2(\hat{A}\hat{B}+\hat{B}\hat{A})$ the symmetric product of two operators.

One should emphasize that in such a situation one does *not* expect time translational invariance to hold, which would manifest itself in, for instance, $A(t)=A_0=\text{const.}$ and $C_{AB}(t_1,t_2)=C_{AB}(t_1-t_2)$. There will be a transient regime in which these relations are violated and depending on the complexity of the system this *non-equilibrium* regime will extend over the whole time axis, in which we would denote it as *quantum aging*, as it can be observed for instance for the universe, which is (most probably) a closed system.

To be concrete we consider the prototype of an interacting quantum systems, the Ising model in a transverse field (TIM) in one dimension defined by the Hamiltonian:

$$H = -\frac{1}{2}\left[\sum_{l=1}^{L-1}\sigma_l^x\sigma_{l+1}^x + h\sum_{l=1}^{L}\sigma_l^z\right] \; , \qquad (3)$$

where $\sigma_l^{x,z}$ are spin-1/2 operators on site l. We consider initial many body states that are eigenstates either of all local σ_l^x or of all local σ_l^z operators. We will mostly be concerned with fully magnetized initial states, either in the x or the z direction, which we denote with $|x\rangle$ and $|z\rangle$, respectively, and which obey $\sigma_l^x|x\rangle=+|x\rangle$ and $\sigma_l^z|z\rangle=+|z\rangle$, respectively.

In passing we note that one obtains the zero temperature *equilibrium* situation by choosing the ground state of the Hamiltonian (3) as the initial state. This ground state has a quantum phase transition at $h=1$ from a paramagnetic ($h>1$) to a ferromagnetic ($h<1$) phase, the latter being indicated by long range order in the x-component, i.e. a non-vanishing expectation value of σ^x. Moreover, non-zero temperature *equilibrium* relaxation has been considered in [6].

The expectation values and correlation functions we are interested in are those that originate from these spin operators σ_l^x and σ_l^z. In order to compute them, we have to express the Hamiltonian (3) in terms of fermion creation (annihilation) operators [7,8] η_q^+ (η_q)

$$H = \sum_q \epsilon_q\left(\eta_q^+\eta_q - \frac{1}{2}\right) \; . \qquad (4)$$

The energy of modes, ϵ_q, $q=1,2,\ldots,L$ are given by the solution of the following set of equations

$$\begin{aligned} \epsilon_q\Psi_q(l) &= -h\Phi_q(l) - \Phi_q(l+1) \; , \\ \epsilon_q\Phi_q(l) &= -\Psi_q(l-1) - h\Psi_q(l) \; , \end{aligned} \qquad (5)$$

and we use free boundary conditions which implies for the components $\Phi_q(L+1) = \Psi_q(0) = 0$. The spin-operators can then be expressed by the fermion operators as

$$\begin{aligned} \sigma_l^x &= A_1 B_1 A_2 B_2 \ldots A_{l-1} B_{l-1} A_l \ , \\ \sigma_l^z &= -A_l B_l \ , \end{aligned} \tag{6}$$

with

$$\begin{aligned} A_i &= \sum_{q=1}^{L} \Phi_q(i)(\eta_q^+ + \eta_q) \ , \\ B_i &= \sum_{q=1}^{L} \Psi_q(i)(\eta_q^+ - \eta_q) \ , \end{aligned} \tag{7}$$

and the time-evaluation of the spin operators follows from that of the fermion operators: $\eta_q^+(t) = e^{it\epsilon_q}\eta_q^+$ and $\eta_q(t) = e^{-it\epsilon_q}\eta_q$.

To calculate different non-equilibrium correlation functions we have developed a systematic method [9] in which the time-dependent contractions are defined by:

$$\begin{aligned} \langle A_l A_k \rangle_t &= \sum_q \cos(\epsilon_q t)\Phi_q(l)\Phi_q(k) \ , \\ \langle A_l B_k \rangle_t &= \langle B_k A_l \rangle_t = i \sum_q \sin(\epsilon_q t)\Phi_q(l)\Psi_q(k) \ , \\ \langle B_l B_k \rangle_t &= \sum_q \cos(\epsilon_q t)\Psi_q(l)\Psi_q(k) \ . \end{aligned} \tag{8}$$

play a central role. For general position of the spin, $l = O(L/2)$, one finds simple formulas for the expectation values and correlation functions involving σ_l^z operators, whereas the calculation of those involving σ_l^x operators is a difficult task and the final result is complicated [10]. However, both the surface-spin auto-correlations and the end-to-end correlations are given in quite simple form, both for the equilibrium [8] and for the non-equilibrium case.

First we study the x-end-to-end correlations defined by

$$C_L^{x,\psi}(t) = \langle \psi_0 | \{ \sigma_1^x(t) \sigma_L^x(t) \}_S | \psi_0 \rangle \ , \tag{9}$$

which contain information about the existence or absence of magnetic order in the x-direction. The single time t at which the correlations between the two spins are measured indicates the age of the system after preparation. For the fully ordered initial state $|\psi_0\rangle = |x\rangle$ we obtain

$$C_L^{x,x}(t) = \langle A_1 A_1 \rangle_t \langle B_L B_L \rangle_t + |\langle A_1 B_L \rangle_t|^2 \ , \tag{10}$$

The first term in the r.h.s. of Eq.(10) is the product of surface magnetizations at the two ends of the chain. Therefore $\lim_{L,t\to\infty} C_L^{x,x}(t) = \overline{m_1}^2$ and the stationary state, starting with $|x\rangle$, has long-range order for $h < 1$ as $\overline{m_1} = 1 - h^2$. Thus the surface order-parameter, $\overline{m_1}$, vanishes continuously at the transition point, $h_c = 1$, with a non-equilibrium exponent, $\beta_1^{ne} = 1$.

The time dependence of the *connected* correlations (generally defined via 2 as $\tilde{C}_{AB}(t_1,t_2) = C_{AB}(t_1,t_2) - A(t_1)B(t_2)$) $\tilde{C}_L^{x,x}(t) = |\langle A_1 B_L \rangle_t|^2$ shows the following features which can be read from fig. 1: 1) They are zero for times smaller than a time $\tau_h(L)$ which is equal to the system size L for $h \geq 1$ and increases monotonically with decreasing h for $h < 1$. 2) At $t = \tau_h(L)$ a jump occurs to a value that decreases algebraically with the system size L:

$$\tilde{C}_{\max}^{x,x}(L) = \tilde{C}_L^{x,x}(t = \tau_h(L)) \propto L^{-a} \ , \tag{11}$$

with $a = 2/3$ for $h = 1$ and $a = 1/2$ for $h > 1$. 3) For $t \geq \tau_h(L)$ the correlations decay slower than exponentially, roughly with a stretched exponential. 4) For $t = 3\tau_h(L)$ again a sudden jump occurs as for $t = \tau_h(L)$ followed by a slightly slower oscillatory decay. 5) This pattern is repeated for time $t = 5\tau_h(L), 7\tau_h(L), \ldots$, but gets progressively smeared out by oscillations.

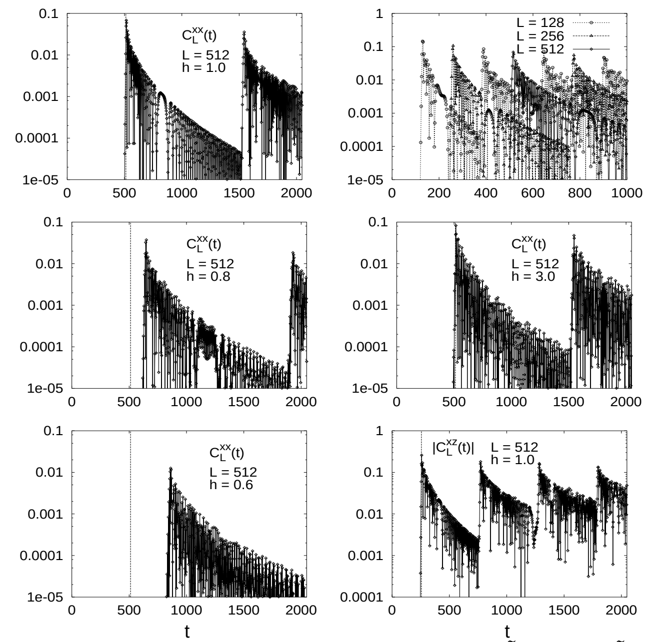

FIG. 1. Connected end-to-end correlations $\tilde{C}_L^{xx}$ (and $\tilde{C}_L^{xz}$ for bottom left) with fixed system size L and field h as a function of the time t calculated with (10) and (12). The left column shows data for decreasing field strength $h = 1.0$, 0.8, 0.6, the broken vertical line is at $t = 512 = L$. The upper right figure shows C_L^{xx} for different system sizes at $H = 1.0$, the middle right plot shows C_L^{xx} at $h = 3.0$ and the lower right figure shows the modulus of C_L^{xz} for $h = 1$. Here the broken vertical line is at $t = 256 = L/2$. For the interpretation see text.

These features can be interpreted as follows: the elementary (tunnel) processes of the quantum dynamics of the Hamiltonian (3) are spin flips induced by the transverse field operator σ_l^z. In this picture two spins can only act coherently and thus give a contribution to the connected correlation function if the information about such a spin flip processes reaches the two spins simultaneously. Feature 1 tells us that signals generated in the center of the system travel with a speed of proportional to

$L/\tau_h(L)$ to the boundary spins and reaches both simultaneously. At this moment $\tilde{C}_L^{x,x}(t)$ jumps to its maximum (see 2). After this, this signal is superposed by other more incoherent signals (see 3). However, the strongest initial signal is reflected at both boundaries and reaches the opposite boundary spins simultaneously again at time $t = 3\tau_h(L)$ (see 4), and so on. Of course more and more incoherent processes occur in the meantime, giving rise to feature 5.

A similar behavior can be observed for the end-to-end correlations when starting with the state $|z\rangle$, which is

$$C_L^{x,z}(t) = \sum_k \left(\langle A_1 B_k\rangle_t \langle B_L A_k\rangle_t - \langle A_1 A_k\rangle_t \langle B_L B_k\rangle_t\right) \ . \tag{12}$$

The only difference to the behavior of $C_L^{x,x}(t)$ reported above is a) its long time limit vanishes for all values of h and b) $\tau_h(L)$, i.e. the earliest time at which the two boundary spins are correlated, is only half as big as in the previous case. Obviously it is easier to generate and to propagate spin flip signals when starting with a z-state.

Next we study the *bulk* behavior of the expectation values and correlations involving σ_l^z operators. We start its non-equilibrium expectation value

$$\begin{aligned} e_l^\psi(t) &= \langle\psi_0|\sigma_l^z(t)|\psi_0\rangle \qquad (13)\\ &= \sum_k \left(\langle A_l B_k\rangle_t \langle B_l A_{i(k)}\rangle_t - \langle A_l A_{i(k)}\rangle_t \langle B_l B_k\rangle_t\right) \ , \end{aligned}$$

with $i(k) = k, (k+1)$ for $\psi = z, (x)$. We note that the equilibrium (i.e. ground state) expectation value, e_l^0, corresponds to the energy-density in the two-dimensional classical Ising model and we use this terminology also in this non-equilibrium situation. For long times the non-equilibrium energy-density approaches a finite limit, $\overline{e}_l^\psi$, which for a bulk spin is a) for the initial state $|\psi_0\rangle = |x\rangle$ given by $\overline{e^x} = h/2$ for $h \leq 1$ and $\overline{e^x} = 1/(2h)$ for $h > 1$. and b) for the initial state $|\psi_0\rangle = |z\rangle$ by $\overline{e^z} = 1/2$ for $h \leq 1$ and $\overline{e^z} = 1 - 1/2h^2$ for $h > 1$. Therefore the analogue to the specific heat $c_v \sim \partial\overline{e^z}/\partial h$ is discontinuous at the transition point. The relaxation of the energy-density to its stationary value is algebraic and follows a $\sim t^{-3/2}$ low for any value of the transverse field, h. At the transition point, $h = 1$, we have the analytical results in terms of the Bessel-function, $J_\nu(x)$: $e_l^\psi(t) = 1/2 \pm J_1(4t)/4t$, where the $+$ $(-)$ sign refer to $\psi = z(x)$.

The two-spin non-equilibrium dynamical and spatial correlations involve contributions from different processes described by the contractions (8) and the corresponding formulas are complicated, therefore they will be presented elsewhere [9]. Here we report on the basic features of the asymptotic behavior of correlations. The two-time correlation function $(t_1 \leq t_2)$,

$$G_l^{z,\psi}(t_1,t_2) = \langle\psi_0|\{\sigma_l^z(t_1)\sigma_l^z(t_2)\}_S|\psi_0\rangle \ , \tag{14}$$

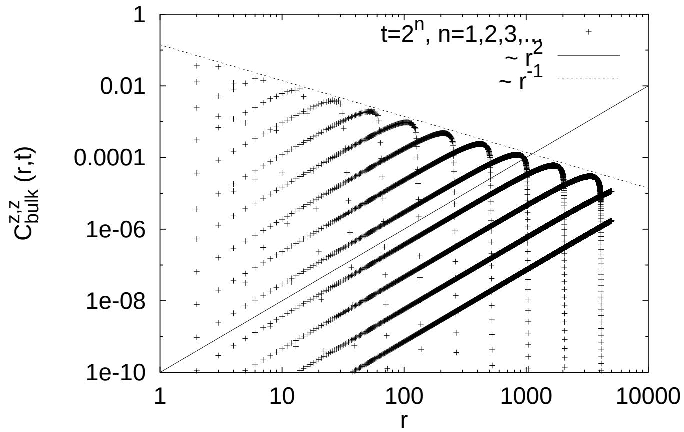

FIG. 2. Connected σ_z correlation function $\tilde{C}^{z,\psi}(r,t)$ at $h = 1$ given by the expression (17) for different times ($t = 2^n$, $n = 1, 2, 3, \dots$ from left to right) in a log-log plot. The two straight lines indicate the initial r^2 dependence of $\tilde{C}^{z,\psi}(r,t)$ for fixed t as well as the r^{-1} dependence of the maximum value at $r = 2t$.

is non-stationary for intermediate times, $t_1/(t_2 - t_1) = O(1)$, which can be read off from our analytical result for the connected bulk auto-correlations at $h = 1$:

$$\tilde{G}_l^{z,\psi}(t_1,t_2) = J_0^2(2t_2 - 2t_1) - \frac{1}{4}[f(t_2 + t_1) \pm g(t_2 - t_1)] \tag{15}$$

where $f(x) = J_2(2x) + J_0(2x)$, $g(x) = J_2(2x) - J_0(2x)$ and the $+$ $(-)$ sign refer to $\psi = x(z)$. Thus we conclude that for intermediate times there is *aging* in the z-component auto-correlation function, contrary to what is reported in [5]. Asymptotically we have $\lim_{t_1\to\infty} G_l^{z,\psi}(t_1,t_2) = (\overline{e}^\psi)^2$, and the connected two-time correlations depends only on the time difference, e.g. for $h = 1$ via (15) $\lim_{t_1\to\infty} \tilde{G}_l^{z,\psi}(t_1,t_2) = J_0^2(2[t_2 - t_1]) - \{J_1'(2[t_2 - t_1])\}^2$. For bulk spins this stationary correlation function decays algebraically as $\sim (t_2 - t_1)^{-2}$ for any value of h.

Next we consider the spatial equal-time correlations

$$C^{z,\psi}(r,t) = \langle\psi_0|\{\sigma_{i-r/2}^z(t)\sigma_{i+r/2}^z(t)\}_S|\psi_0\rangle \ , \tag{16}$$

where $i = L/2$ in a finite system. For long times they approach the stationary limit, $\lim_{t\to\infty} C^{z,\psi}(r,t) = (\overline{e}^\psi)^2$, the same as for the auto-correlation function. For the connected correlation function $\tilde{C}^{z,\psi}(r,t) = C^{z,\psi}(r,t) - e_l^\psi(t)e_{l+r}^\psi(t)$ we can derive an analytic expression at the transition point, $h = 1$ in the limit $L \to \infty$

$$\begin{aligned} \tilde{C}^{z,\psi}(r,t) &= \left[\frac{r}{2t}J_{2r}(4t)\right]^2 \qquad (17)\\ &\quad - \frac{r^2-1}{4t^2}J_{2r+1}(4t)J_{2r-1}(4t) \ , \end{aligned}$$

which is valid both for $|\psi_0\rangle = |x\rangle$ and $|\psi_0\rangle = |z\rangle$. In fig. 2 we show the r-dependence of $\tilde{C}^{z,z}(r,t)$ for various times

t. We see that for fixed time t the correlations increase proportional to r^2 for distances $r \leq t$ to a maximum value $\tilde{C}^{z,z}_{\max}(t)$ at $r = 2t$, which decreases with time proportional to t^{-1}. For distances larger than $r = 2t$ they drop rapidly, faster than exponentially, to zero.

The latter two features correspond perfectly to what we observed also for the z-end-to-end correlations, see eq(12): spins that are separated by a distance r can only be correlated after the first signal from spin flip processes in between them reach simultaneously the two spins, i.e. for times t larger than $r/2$ (for $h = 1$ and $|\psi_0\rangle = |z\rangle$. The first feature, that correlations for distances smaller than $2t$ are diminished only algebraically rather than via a stretched exponential in the case of end-to-end correlations, is new and characteristic for bulk spins. For $r \leq 2t$ the correlation function $\tilde{C}^{z,\psi}(r,t)$ obeys the characteristic scaling form

$$\tilde{C}^{z,\psi}(r,t) = t^{-1} g(r/t) \qquad (18)$$

with $g(x) \propto x^2$ for $x \ll 1$. The scaling parameter r/t appearing in the scaling function $g(x)$ is reminiscent of the fact that space and time scales are connected linearly at the critical point in the transverse Ising chain since the dynamical exponent is $z = 1$. Away from the critical point we have to evaluate our expressions [9] for $\tilde{C}^{z,\psi}(r,t)$ numerically for finite but large ($L = 512$) system sizes. Essentially we observe the same scenario as at the critical point, the only difference being that the general functional dependency is superposed by strong oscillations. Moreover, starting with $|\psi_0\rangle = |x\rangle$ instead of $|z\rangle$ changes the correlations only by a constant factor.

We collect now our results for the maximum value for connected spin-spin correlations since they decay algebraically with various new exponents. we define ourselves to $h \geq 1$ since here the time τ_h of maximum correlation is fixed, whereas for $h < 1$ the value of τ_h depends on h and has to be determined numerically that renders the precise determination of the decay exponents difficult. We define the ratio $\alpha = t/L$ and $\alpha_{\max} = \tau_h(L)/L$ and consider equal time correlations for fixed values of α. In the picture of a propagating front, that separates a region in the space-time diagram for the chain in which spins are uncorrelated from a region in which they are correlated, one observes quasi long range correlations *on the front*, the latter being defined by the ratio $t/L = \alpha_{\max}$. For distances smaller than the distance of maximum correlation or times larger than τ_h the correlations decay slower than exponential in time, e.g. algebraically for bulk spins ($\tilde{C}^{z\psi}(t, r = \text{fixed}) \sim t^{-2}$)). When we vary both space and time with fixed ratio t/L or t/r we get power laws, as long as we stay *behind* the front (i.e. $t \geq \tau_h$). For $\alpha > \alpha_{\max}$ we observe again power laws, but with different exponents; they are listed in table 1.

To conclude we studied a novel type of dynamically produced long range correlations in a quantum relaxation process in a quantum spin chain. Starting with a homogeneous initial state the quantum mechanical time evolution according to the Schrödinger equation drives the system into a stationary state, which has algebraically decaying time-dependent autocorrelations but no critical fluctuations. However, *during* the relaxation process spin-spin correlation build up upon arrival of a front of coherent signals, which afterwards decay algebraically in the bulk. *On* the front and behind it for fixed ratio of space and time scales one observes quasi long range order. This does *not* depend on any external parameter like the transverse field. This type of algebraic correlation needs not to be triggered by some tuning parameter and is therefore reminiscent of phenomena in self organized criticality [11]. The scenario we have reported here is a result of quantum interference and one may expect that a similar one holds for other quantum systems, too. At this point one should mention the possibility of coarsening in quantum systems as for instance reported in [12], which is different from the scenario we have reported here.

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		$h=1$		$h>1$	
	$\alpha_{\max}$	$\alpha{=}\alpha_{\max}$	$\alpha{>}\alpha_{\max}$	$\alpha{=}\alpha_{\max}$	$\alpha{>}\alpha_{\max}$
$\tilde{C}^{xx}_L(t{=}\alpha L)$	1	$L^{-2/3}$	L^{-1}	$L^{-1/2}$	L^{-1}
$\tilde{C}^{xz}_L(t{=}\alpha L)$	1/2	$L^{-1/4}$	$L^{-1/2}$	–	–
$C^{z\psi}_L(t{=}\alpha L)$	1/2	$L^{-5/4}$	L^{-1}	$L^{-5/8}$	L^{-1}
$\tilde{C}^{z\psi}(t{=}\alpha r)$	1/2	$r^{-4/3}$	r^{-1}	$r^{-2/3}$	r^{-1}

TABLE I

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